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The initial boundary-value mixed problems for elastic half-plane with the conditions of contact friction

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Abstract

The exact solutions of nonstationary contact problems of elastodynamics for a half-plane with the dry and viscous friction in the contact zone having the contact edge point moving with arbitrary variable velocity along the boundary of a half-plane are obtained in a closed form. A new method of solution based on the use of Radon transform is used.

Keywords: Elastodynamics; Contact problems; Method for solution

1. Introduction

The initial boundary-value problems for an elastic half-plane considered in this paper, are the examples of the nonstationary contact problems of elastodynamics with the dry or viscous friction in a contact zone. The nonstationary contact problems of elastodynamics were studied by many authors. Considerable progress has been made in these problems with frictionless contact (Flitman, 1959; Bedding and Willis, 1973, 1976; Willis, 1973, 1989; Robinson and Thompson, 1974; Brock, 1976, 1977, Brock, 1978, 1979; Cherepanov, 1979; Georgiadis and Barber, 1993, and others).

A nonstationary (transient) contact problem with friction (Coulomb's dry friction) was first considered by Brock (1981). In what follows, the dynamic problems with friction were investigated in the works of Brock (1993), Brock and Georgiadis (1994) and Georgiadis et al. (1995). In doing so, practically all the problems considered in these works, are self-similar (automodeling). The investigation of automodeling contact problems gives a considerable useful information on the elastodynamic fields in the contact zone (see Brock, 1993, Brock and Georgiadis, 1994; Georgiadis et al., 1995), but, nevertheless, some elastodynamic contact effects lie outside the framework of automodeling description.

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From the viewpoint of mathematical analysis, no automodeling contact problems with friction and moving with arbitrary variable velocity edge of the contact zone are difficult even in the so-called canonical case where the contact region is semi-infinite. Application of the traditional methods to solve these problems meets the great difficulties and, for this reason, the exact solutions of the problems of this class are still absent.

In this paper, the nonstationary contact problems for an elastic half-plane with the dry or viscous friction in a contact zone are solved by the new method (Shmegera, 1997, 1998). The method is based on the use of Radon transform described briefly in Appendix A.

2. Statement of problems

Consider a homogeneous, isotropic and linearly elastic half-plane $y < 0$ and $-\infty < x < \infty$ under plane-strain condition, where (x, y) are Cartesian coordinates. Let for $t > 0$, where t is the time, on the boundary of half-plane the following boundary conditions for the components of vector displacement $\mathbf{w} = \{u, v\}$ ($u(x, y, t)$ and $v(x, y, t)$ are the projections of $\mathbf{w}$ on the x-axis and y-axis, respectively) and the components $\sigma_y(x, y, t)$ and $\tau_{xy}(x, y, t)$ of stress tensor are given. For $x > l(t)$ and $y = 0$, where $l(t)$ is an arbitrary bounded function of time, the normal and tangential loads are applied. We can assume, without loss of generality, that

$$\sigma_y(x, 0, t) = \tau_{xy}(x, 0, t) = 0 \quad x > l(t), t > 0. \tag{1}$$

For $x < l(t)$ and $y = 0$ the boundary of half-plane is interacting with a rigid body and the following contact conditions occur:

1. the contact with Coulomb (dry) friction:

$$v(x, 0, t) = v_0(x, t), \quad \tau_{xy}(x, 0, t) = -k\sigma_y(x, 0, t), \quad \sigma_y(x, 0, t) \leq 0, \qquad x < l(t), t > 0; \tag{2}$$

2. the contact with viscous friction:

$$v(x, 0, t) = v_0(x, t), \quad \tau_{xy}(x, 0, t) = f\left[\dot{u}(x, 0, t) + w_0(x, t)\right], \quad \sigma_y(x, 0, t) \leq 0, \qquad x < l(t), t > 0. \tag{3}$$

Here k and f are the coefficients of dry and viscous friction, respectively, $\dot{u}_k = \partial u/\partial t$, $w_0(x, t)$ is a mass velocity of rigid body along the half-plane boundary ($w_0 \leq \mathrm{d}l/\mathrm{d}t$),

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x},$$

$$\sigma_y = \mu\left[\frac{c_1^2}{c_2^2}\Delta\phi - 2\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right)\right],$$

$$\tau_{xy} = \mu\left[\Delta\psi - 2\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \phi}{\partial x \partial y}\right)\right], \tag{4}$$

where μ is a shear modulus, c_1 and c_2 are the longitudinal and shear wave speeds, respectively, and $\phi(x, y, t)$ and $\psi(x, y, t)$ are the displacement potentials satisfying the wave equations

$$\Delta\phi = \frac{1}{c_1^2}\ddot{\phi}, \quad \Delta\psi = C\frac{1}{c_2^2}\ddot{\psi} \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; (\ddot{\ }) \equiv \frac{\partial^2}{\partial t^2}. \tag{5}$$

The initial condition are zero:

$$\mathbf{w}(x, y, t) = \dot{\mathbf{w}}(x, y, t) = 0, \quad t < 0. \tag{6}$$

The displacement must be bounded and continuous at the vicinity of point $x = l(t)$ and $y = 0$ where the type of boundary conditions changes, i.e.

$$\mathbf{w}(x, y, t) \simeq \mathbf{a}(t) + O(r^{\epsilon}), \quad r \to 0 \left(r = \sqrt{(x - l(t))^2 + y^2} \right). \tag{7}$$

Here $\mathbf{a}(t)$ is a certain bounded function of time and

$$\begin{aligned} &\epsilon > 0 && \text{for } \mathrm{d}l/\mathrm{d}t = 0, \\ &\epsilon \geq 1/2 && \text{for } \mathrm{d}l/\mathrm{d}t \neq 0. \end{aligned} \tag{7a}$$

The condition (7) is equivalent (see e.g. Poruchikov, 1986) to the condition of the nonnegativity and boundedness of the energy flux at the point $x = l(t)$ and $y = 0$ (the edge of the contact zone). This condition is necessary for the uniqueness of solution.

We will seek the solutions of wave equations in the form of continuous (integral) superposition of arbitrary plane waves (see Appendix A, and also Shmegera, 1997, 1998)

$$\phi(x, y, t) = \mathrm{Re}\frac{1}{2\pi i}\int_{\Gamma} F_1(z_1(x, y, t, c), c)\,\mathrm{d}c,$$

$$\psi(x, y, t) = \mathrm{Re}\frac{1}{2\pi i}\int_{\Gamma} F_2(z_2(x, y, t, c), c)\,\mathrm{d}c, \tag{8}$$

where $F_j(z_j)$ (here and everywhere below $j = 1, 2$) are arbitrary, twice differentiable (or analytic, if z_j are of complex) functions. The functions z_j are

$$z_j = \xi + i\eta_j, \quad \xi = x - ct, \quad \eta_j = \gamma_j y, \quad \gamma_j = \sqrt{1 - c^2/c_j^2}, \qquad i = \sqrt{-1}. \tag{9}$$

The branches of radicals $(1 - c^2/c_j^2)^{1/2}$ in the complex plane c with the cuts $(-\infty, c_j)$ and (c_j, ∞) along

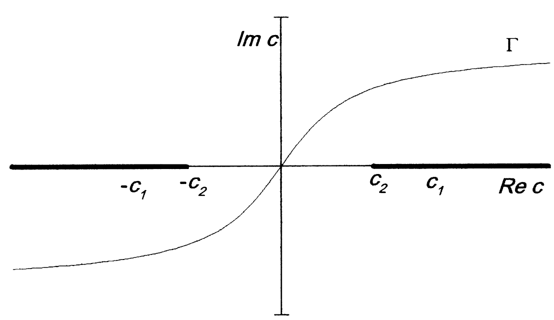

Fig. 1. The contour of integration Γ and the cuts in the complex plane c.

the axis $\operatorname{Im} c = 0$ (Fig. 1), are fixed by the conditions $(1 - c^2/c_j^2) > 0$ for $\operatorname{Im} c > 0$. The contour Γ is shown in Fig. 1.

The displacements and stresses (4) in terms of Eqs. (8) and (9) are written as

$$u = \operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \left[F_1'(z_1(c), c) + i\gamma_2(c)F_2'(z_2(c), c)\right] \mathrm{d}c,$$

$$v = \operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \left[i\gamma_1 F_1'(z_1(c), c) - F_2'(z_2(c), c)\right] \mathrm{d}c,$$

$$\sigma_y = -\mu\operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \left[\gamma(c)F_1''(z_1(c), c) + 2i\gamma_2(c)F_2''(z_2(c), c)\right] \mathrm{d}c,$$

$$\tau_{xy} = \mu\operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \left[2i\gamma_1(c)F_1''(z_1(c), c) - \gamma(c)F_2''(z_2(c), c)\right] \mathrm{d}c, \tag{10}$$

where $F_j' = \partial F_j/\partial z_j$, $F_j'' = \partial^2 F_j/\partial z_j^2$, and

$$\gamma = 1 + \gamma_2^2. \tag{11}$$

It follows from the representations (10) that to solve the problems (1)–(3) it is sufficient to find the functions F_j.

3. Problem with dry friction: general solution

Consider first the problem with the boundary conditions (1) and (2). Substituting Eq. (10) (after differentiating the condition for $v(x, 0, t)$ in Eq. (2) with respect to x) into these conditions, we rewrite them in the form

$$\operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \Sigma\big(\xi(x, t, c), c\big)\, \mathrm{d}c = 0, \quad x > l(t),\ t > 0,$$

$$\operatorname{Re}\frac{1}{2\pi i}\int_\Gamma T\big(\xi(x, t, c), c\big)\, \mathrm{d}c = 0, \quad x > l(t),\ t > 0,$$

$$\operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu R(c)}\left[n_1(c)\Sigma\big(\xi(x, t, c), c\big) + q(c)T\big(\xi(x, t, c), c\big)\right] \mathrm{d}c = v_0'(x, t), \quad x < l(t),\ t > 0,$$

$$\operatorname{Re}\frac{1}{2\pi i}\int_\Gamma \left[T\big(\xi(x, t, c), c\big) + k\Sigma\big(\xi(x, t, c), c\big)\right] \mathrm{d}c = 0, \quad x < l(t),\ t > 0, \tag{12}$$

where $v_0' = \partial v_0/\partial x$ and the following notations are introduced

$$\Sigma(\xi) = -\mu\left[\gamma F_1''(\xi) + 2i\gamma_2 F_2''(\xi)\right], \qquad T(\xi) = \mu\left[2i\gamma_1 F_1''(\xi) - \gamma F_2''(\xi)\right], \tag{13}$$

$$R = \gamma^2 - 4\gamma_1\gamma_2, \quad q = \gamma - 2\gamma_1\gamma_2, \quad n_1 = i\gamma_1\left(1 - \gamma_2^2\right). \tag{14}$$

Applying the Radon transform in the form (see Appendix A, formula (A18))

$$F(\xi, c) = \frac{\partial}{\partial \xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, t)\delta(x - ct - \xi)\,\mathrm{d}x\,\mathrm{d}t, \tag{15}$$

to the right and left sides of Eqs. (12) and taking into account the properties of Radon transform carrier (see the formulae (A20)–(A22) in Appendix A), we obtain

$$\mathrm{Re}\,\Sigma(\xi) = 0, \quad \xi < l_* - ct, \xi > l_*, \tag{16}$$

$$\mathrm{Re}\,T(\xi) = 0, \quad \xi < l_* - ct, \xi > l_*, \tag{17}$$

$$\mathrm{Re}\frac{1}{\mu R}\left[n_1\Sigma(\xi) + qT(\xi)\right] = V_0'(\xi), \quad l_* - ct < \xi < l_*, \tag{18}$$

$$\mathrm{Re}\left[k\Sigma(\xi) + T(\xi)\right] = 0, \quad l_* - ct < \xi < l_*. \tag{19}$$

Here $l_* = l(t_*)$, where t_* is a solution of equation (see Appendix A)

$$\xi + ct_* - l(t_*) = 0, \tag{20}$$

and

$$V_0'(\xi) = \frac{\partial}{\partial \xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_0'\left(x, t'\right)H(t')H(t - t')H(l(t') - x)\delta\left(x - ct' - \xi\right)\,\mathrm{d}x\,\mathrm{d}t'. \tag{21}$$

The Heaviside functions $H(\cdots)$ are introduced to emphasize that the carrier of the function $v_0(x, t)$ is bounded.

The relations (16)–(19) can be treated as a system of boundary problems of Riemann–Hilbert type for the functions $\Sigma(\xi)$ and $T(\xi)$. With these known functions, the functions $F_j''(\xi)$ can be determined from Eq. (13) as

$$F_j''(\xi) = \Lambda_j(\xi), \tag{22}$$

where

$$\Lambda_1(\xi) = \frac{1}{\mu R}\left[-\gamma\Sigma(\xi) + 2i\gamma_2 T(\xi)\right],$$

$$\Lambda_2(\xi) = -\frac{1}{\mu R}\left[2i\gamma_1\Sigma(\xi) + \gamma T(\xi)\right]. \tag{23}$$

Then the function $F_j''(z_j)$ can be found with the aid of Cauchy integral for the half-plane $\mathrm{Im}\,z_j > 0$

$$F_j''(z_j) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Lambda_j(\xi)}{\xi - z_j}\,\mathrm{d}\xi, \tag{24}$$

and next, the expressions for the stresses can be obtained from Eq. (10).

The system of boundary-value problems (16)–(19) reduces easily to a boundary-value problem for one unknown function. Multiplying the condition (16) by k and adding the conditions (16) and (17), we write them in the form

$$\mathrm{Re}\big[k\Sigma(\xi) + T(\xi)\big] = 0, \quad \xi < l_* - ct, \xi > l_*. \tag{25}$$

The conditions (19) and (25) can be considered as a problem for the expression $k\Sigma(\xi) + T(\xi)$. The solution of this problem is given by Cauchy type integral (see e.g. Gakhov, 1966) and in the class of functions vanishing at infinity has the form

$$k\Sigma(\xi) + T(\xi) = 0. \tag{26}$$

Using Eq. (26) to eliminate the function $T(\xi)$ from Eq. (18), we obtain

$$\mathrm{Re}\left\{\frac{1}{\mu R}\big[(n_1 - kq)\Sigma(\xi)\big]\right\} = V_0'(\xi), \quad l_* - ct < \xi < l_*. \tag{27}$$

The conditions (27) and (16) represent a Riemann–Hilbert problem for the function $\Sigma(\xi)$. It is more convenient to rewrite this problem as a Riemann problem (Gakhov, 1966) or, as a problem of conjugation (Muskhelishvili, 1953a)

$$\Sigma^+(\xi) - \Sigma^-(\xi) = 0, \quad \xi < l_* - ct, \xi > l_*,$$

$$\Sigma^+(\xi) - \frac{R}{\bar{R}}\frac{\bar{n}_1 - k\bar{q}}{n_1 - kq}\Sigma^-(\xi) = \frac{2\mu R}{n_1 - kq}V_0'(\xi), \quad l_* - ct < \xi < l_*, \tag{28}$$

in this case and elsewhere, the notations with overbar are the complex conjugation. The solution of problem (28) (vanishing at infinity) can be found using the known formulae for the Riemann problem with a discontinuous coefficient (Gakhov, 1966). Thus, we obtain for $z_j = \xi + i0$

$$\Sigma^+(\xi) = \frac{1}{\pi i}\frac{\mu R}{Q}\left[G(\xi)\int_{l_*-ct}^{l_*} G^{-1}(\xi')\frac{V_0'(\xi')}{\xi' - \xi}\,\mathrm{d}\xi' + \pi i V_0'(\xi)\right]. \tag{29}$$

Here

$$G(\xi) = \left(\frac{l_* - \xi}{l_* - ct - \xi}\right)^{\alpha}, \quad \alpha = -\frac{1}{2\pi i}\ln\beta, \tag{30}$$

where

$$\beta = \frac{R}{\bar{R}}\frac{\bar{Q}}{Q}, \quad Q = n_1 - kq, \tag{31}$$

provided that

$$-\pi < \arg\beta < \pi. \tag{32}$$

The expression for $T(\xi)$ follows from Eq. (26).

Now, using the formulae (29), (26), (23) and (10), we obtain the expressions for stresses in the half-plane

$$\begin{bmatrix} \sigma_y(x, y, t) \\ \sigma_{xy}(x, y, t) \end{bmatrix} = \frac{\mu}{2\pi^2} \mathrm{Re} \int_\Gamma \frac{1}{Q(c)} \left\{ \begin{bmatrix} -\gamma^2(c) - 2ki\gamma(c)\gamma_2(c) \\ 2i\gamma(c)\gamma_1(c) - 4k\gamma_1(c)\gamma_2(c) \end{bmatrix} P(z_1(x, y, t, c), c) \right.$$
$$\left. + \begin{bmatrix} 4\gamma_1(c)\gamma_2(c) + 2ki\gamma(c)\gamma_2(c) \\ -2i\gamma(c)\gamma_1(c) + k\gamma^2(c) \end{bmatrix} P(z_2(x, y, t, c), c) \right\} \mathrm{d}c. \tag{33}$$

Here

$$P(z_j) = G(z_j) \int_{l_*-ct}^{l_*} G^{-1}(\xi) \frac{V_0'(\xi)}{\xi - z_j} \, \mathrm{d}\xi,$$

where G is of the form (30).

The expressions (33), (20) and (21) determine entirely the stresses in the half-plane provided that the function $l(t)$ (the edge of contact zone) is known. If, however $l(t)$ is unknown, then it is necessary to complement the expressions (33), (20) and (21) with an equation for $l(t)$. The way of obtaining such an equation is considered in the following section.

4. The stresses in the contact zone

Now, we consider the stresses in the contact zone for $x < l(t)$ and $y = 0$. Taking into account Eq. (10) and the notations (22) and (23), the expressions for stresses on the boundary of half-plane can be written as

$$\begin{bmatrix} \sigma_y(x, 0, t) \\ \tau_{xy}(x, 0, t) \end{bmatrix} = \mathrm{Re} \frac{1}{2\pi i} \int_\Gamma \begin{bmatrix} \Sigma(\xi(x, t, c), c) \\ T(\xi(x, t, c), c) \end{bmatrix} \mathrm{d}c. \tag{34}$$

Substituting the expressions for $\Sigma(\xi)$, Eq. (29), and for $T(\xi)$, Eq. (26), into Eq. (34) and taking into account Eqs. (30) and (9), we obtain

$$\begin{bmatrix} \sigma_y(x, 0, t) \\ \tau_{xy}(x, 0, t) \end{bmatrix} = -\frac{\mu}{2\pi^2} \begin{bmatrix} 1 \\ -k \end{bmatrix} \mathrm{Re} \int_\Gamma \frac{R(c)}{Q(c)} \left\{ \left(\frac{l_* - x + ct}{l_* - x} \right)^{\alpha(c)} \right.$$
$$\left. \times \int_{l_*-ct}^{l_*} \left(\frac{l_* - \xi'}{l_* - ct - \xi'} \right)^{-\alpha(c)} \frac{V_0'(\xi', c)}{\xi' - x + ct} \, \mathrm{d}\xi' + \pi i V_0'(x, t, c) \right\} \mathrm{d}c, \tag{35}$$

where $R(c)$, $Q(c)$ and $\alpha(c)$ have the form (see Eqs. (14), (30) and (31))

$$R(c) = \left(2 - c^2/c_2^2\right)^2 - 4\sqrt{1 - c^2/c_1^2}\sqrt{1 - c^2/c_2^2}, \tag{36}$$

$$Q(c) = ic^2 c_2^{-2} \sqrt{1 - c^2/c_1^2} - k\left[2 - c^2/c_2^2 - 2\sqrt{1 - c^2/c_1^2}\sqrt{1 - c^2/c_2^2}\right], \tag{37}$$

$$\alpha(c) = -\frac{1}{2\pi i} \ln \frac{R(c)}{\bar{R}(c)} \frac{\bar{Q}(c)}{Q(c)}. \tag{38}$$

The integrands of Eq. (35) have the branch points $\pm c_2$, $\pm c_1$, c_* and c_{**}, where c_* and c_{**} are the solutions of following equations

$$l_*(c_*) - x + c_* t = 0, \qquad l_*(c_{**}) - x = 0. \tag{39}$$

These integrands are analytic in the c-plane outside the cuts $(-\infty, -c_j)$, (c_j, ∞), (c_*, ∞) and (c_{**}, ∞) along the axis $\operatorname{Im} c = 0$ and with a finite number of poles being excluded. Assume that these poles lie on the axis $\operatorname{Im} c = 0$. Transforming the contour Γ along the axis $\operatorname{Im} c = 0$ and dividing the interval $(-\infty, \infty)$ into $(0, \pm c_2)$, $(\pm c_2, \pm c_1)$ and $(\pm c_1, \pm\infty)$, we write the expression (35) as

$$\begin{aligned}
\begin{bmatrix} \sigma_y(x, 0, t) \\ \tau_{xy}(x, 0, t) \end{bmatrix} = &-\frac{\mu}{2\pi^2}\begin{bmatrix} 1 \\ -k \end{bmatrix}\operatorname{Re}\Bigg\{\left(\int_{-c_2}^{0} + \int_{0}^{c_2}\right)\frac{R(c)}{Q(c)}\Bigg[\left(\frac{l_* - x + ct}{l_* - x}\right)^{\Theta(c)} \\
&\times \int_{l_*-ct}^{l_*}\left(\frac{l_* - \xi'}{l_* - ct - \xi'}\right)^{-\Theta(c)}\frac{V_0'(\xi', c)}{\xi' - x + ct}\,\mathrm{d}\xi' + \pi i V_0'(x, t, c)\Bigg] \\
&+ \left(\int_{-c_1}^{-c_2} + \int_{c_2}^{c_1}\right)\frac{R_1(c)}{Q_1(c)}\Bigg[\left(\frac{l_* - x + ct}{l_* - x}\right)^{\omega(c)}\int_{l_*-ct}^{l_*}\left(\frac{l_* - \xi'}{l_* - ct - \xi'}\right)^{-\omega(c)} \\
&\times \frac{V_0'(\xi', c)}{\xi' - x + ct}\,\mathrm{d}\xi' + \pi i V_0'(x, t, c)\Bigg] \\
&+ \left(\int_{-\infty}^{-c_1} + \int_{c_1}^{\infty}\right)\frac{R_2(c)}{Q_2(c)}\Bigg[\int_{l_*-ct}^{l_*}\frac{V_0'(\xi', c)}{\xi' - x + ct}\,\mathrm{d}\xi' + \pi i V_0'(x, t, c)\Bigg]\Bigg\}\,\mathrm{d}c,
\end{aligned} \tag{40}$$

where

$$\Theta(c) = \frac{1}{\pi}\arctan\frac{c_2^2 c^2\sqrt{1 - c^2/c_1^2}}{k\left(2 - c^2/c_2^2 - 2\sqrt{1 - c^2/c_1^2}\sqrt{1 - c^2/c_2^2}\right)}, \quad 0 < \Theta < \frac{1}{2} \text{ for } kn_1(c)q(c) > 0,$$

$$\omega(c) = \Theta(c) - \frac{1}{\pi}\arctan\frac{4\sqrt{1 - c^2/c_1^2}\sqrt{c^2/c_2^2 - 1}}{\left(2 - c^2/c_2^2\right)^2}, \quad 0 < \omega < \frac{1}{2},$$

$$R_1(c) = \left(2 - c^2/c_2^2\right)^2 - 4i\sqrt{1 - c^2/c_1^2}\sqrt{c^2/c_2^2 - 1},$$

$$R_2(c) = \left(2 - c^2/c_2^2\right)^2 + 4\sqrt{c^2/c_1^2 - 1}\sqrt{c^2/c_2^2 - 1},$$

$$Q_1(c) = ic_2^{-2}c^2\sqrt{1 - c^2/c_1^2} - k\left(2 - c^2/c_2^2 - 2i\sqrt{1 - c^2/c_1^2}\sqrt{c^2/c_2^2 - 1}\right),$$

$$Q_2(c) = -c_2^{-2}c^2\sqrt{c^2/c_1^2-1} - k\left(2 - c^2/c_2^2 + 2\sqrt{c^2/c_1^2-1}\sqrt{c^2/c_2^2-1}\right). \tag{41}$$

The values of arctangents in Eq. (41) are selected with account for the conditions (7) and (7a).

In the case when a geometry of rigid body is such that the function $l(t)$ (the edge of the contact zone) being known, the expressions (40), (20) and (21) entirely determine the stresses in the contact zone. In the case where the rigid body has a differentiable boundary in the edge of the contact zone, the function $l(t)$ is unknown. The equation for $l(t)$ can be obtained (as this is made in the analogous problems of elastostatics, see e.g. Galin, 1953) from the following obvious condition

$$\sigma_y(x, 0, t)|_{x=l(t)} = 0, \tag{42a}$$

which, taking into account (40) can be represented in the form

$$\mathrm{Re}\int_{-\infty}^{\infty}\frac{R(c)}{Q(c)}\left\{\left(\frac{l_*-l(t)+ct}{l_*-l(t)}\right)^{\alpha(c)}\times\int_{l_*-ct}^{l_*}\left(\frac{l_*-\xi'}{l_*-ct-\xi'}\right)^{-\alpha(c)}\frac{V_0'(\xi',c)}{\xi'-l(t)+ct}\,\mathrm{d}\xi' + \pi i V_0'(l(t), t, c)\right\}\mathrm{d}c = 0, \tag{42b}$$

where l_* and l are related by Eq. (20). A simple analysis of the left-hand side of Eq. (42b) shows that this equation has a solution, i.e., the stresses are bounded (in fact, zero) at the point $x = l(t)$, only in the case where $\dot{l}(t) < c_{\mathrm{R}}$ or $\dot{l}(t) > c_1$ ($\dot{l}(t) = \mathrm{d}l/\mathrm{d}t$; c_{R} is the Rayleigh speed: $R(c_{\mathrm{R}}) = 0$). This means that σ_y and τ_{xy} are unbounded for $c_{\mathrm{R}} < \dot{l}(t) < c_1$ provided that $kn_1q > 0$ for $c_{\mathrm{R}} < c < c_2$. In this case, $l(t)$ can be found from the expression for $v_0'(x,t)$ in Eq. (12), which with account for Eqs. (26) and (31) can be written in the form

$$\mathrm{Re}\frac{1}{2\pi i}\int_{\Gamma}\frac{Q(c)}{\mu R(c)}\Sigma(x, t, c)\,\mathrm{d}c = v_0'(x, t). \tag{43a}$$

Integrating Eq. (43a) with respect to x from $-\infty$ to $l(t)$ and assuming that $v_0(x, t)\to 0$ for $x\to-\infty$, we obtain the following equation for $l(t)$ in the case $c_{\mathrm{R}} < \dot{l}(t) < c_1$

$$-\frac{1}{2\pi^2}\int_{-\infty}^{l(t)}\mathrm{Re}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\left(\frac{l_*-l(t)+ct}{l_*-l(t)}\right)^{\alpha(c)}\times\int_{l_*-ct}^{l_*}\left(\frac{l_*-\xi'}{l_*-ct-\xi'}\right)^{-\alpha(c)}\frac{V_0'(\xi',c)}{\xi'-l(t)+ct}\,\mathrm{d}\xi' + \pi i V_0'(l(t), t, c)\right\}\mathrm{d}c\,\mathrm{d}x = v_0(l(t), t). \tag{43b}$$

This condition has the obvious kinematic sense. Note that Eqs. (42b) and (43b), and also Eq. (20) simplify essentially in the case where $\dot{l}(t) = \text{constant}$ (e.g., in the self-similar (automodeling) case).

Now, we derive the asymptotic expressions for stresses at the point $x = l(t)$. Let $l_* = l(t_*) \simeq l(t) - \dot{l}(t)(t - t_*)$ for $x\to l(t)$. Then from Eq. (20), we have

$$t_* \simeq t - \frac{x - l(t)}{c - \dot{l}(t)}, \quad l(t_*) \simeq x - \frac{c(x - l(t))}{c - \dot{l}(t)}. \tag{44}$$

Substituting these asymptotic expressions into Eq. (35) or (40), we find for $x\to l(t) - 0$

$$\begin{bmatrix}\sigma_y(x,0,t)\\ \tau_{xy}(x,0,t)\end{bmatrix} \simeq -\frac{\mu}{\pi^2}\begin{bmatrix}1\\ -k\end{bmatrix}\int_{\dot{l}(t)}^{\infty} \operatorname{Re}\left\{\frac{R(c)}{Q(c)}\left(\frac{\dot{l}(t)-c}{\dot{l}(t)-x}t\right)^{\alpha(c)}\right.$$
$$\left.\times\int_{l(t)-ct}^{l(t)}\left(\xi'+ct-l(t)\right)^{\alpha(c)-1}\left(\xi'-l(t)\right)^{-\alpha(c)}V_0'\left(\xi',c\right)\mathrm{d}\xi'+\pi i V_0'(l(t),t,c)\right\}\mathrm{d}c. \tag{45}$$

It follows from Eqs. (45), (42a) and (42b)) that, for example, in the particular case where the rigid body (punch) has a differentiable boundary at the point of $x = l(t)$, the stresses have the following behavior for $x \rightarrow l(t)$

$$\sigma_y \sim \tau_{xy} \sim \begin{cases} 0 & \text{for } \dot{l}(t) < c_R \\ (l(t)-x)^{-\Theta(\dot{l}(t))} & \text{for } c_R < \dot{l}(t) < c_2 \\ (l(t)-x)^{-\omega(\dot{l}(t))} & \text{for } c_2 < \dot{l}(t) < c_1 \\ v_0'(x,t)|_{x=l(t)} & \text{for } c_1 < \dot{l}(t) < \infty \quad \left(v_0' = \partial v_0/\partial x\right), \end{cases} \tag{46}$$

where $\Theta(\dot{l}(t))$ and $\omega(\dot{l}(t))$ are of the form $\Theta(c)$ and $\omega(c)$ from Eq. (41) for $c = l(t)$ and it is accepted that $kn_1q > 0$ for $c_R < c < c_2$.

5. Problem with viscous friction: general solution

Now, we consider the problem with the more complicated boundary conditions (1) and (3). These conditions relate the case of contact with viscous friction. Substituting Eq. (10) into Eqs. (1) and (3) and applying the Radon transform (15) to the obtained expressions, we have

$$\operatorname{Re}\Sigma(\xi) = \operatorname{Re}T(\xi) = 0, \quad \xi < l_* - ct,\ \xi > l_*,$$

$$\operatorname{Re}\frac{1}{\mu R}\left[n_1\Sigma(\xi) + qT(\xi)\right] = V_0'(\xi), \quad l_* - ct < \xi < l_*,$$

$$\operatorname{Re}\left[\frac{fqc}{\mu R}\Sigma(\xi) + \left(1 - \frac{fn_2c}{\mu R}\right)T(\xi)\right] = fW_0(\xi), \quad l_* - ct < \xi < l_*, \tag{47}$$

where $\Sigma(\xi)$, $T(\xi)$, R, n_j and q are determined in Eqs. (13) and (14), $V_0'(\xi)$ has the form (21), l_* is determined in Eq. (20), and

$$W_0(\xi) = \frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_0\left(x,t'\right)H(t')H(t-t')H(l(t')-x)\delta\left(x-ct'-\xi\right)\mathrm{d}x\,\mathrm{d}t'. \tag{48}$$

The relations (47) can be regarded as a set of Riemann–Hilbert problems for functions $\Sigma(\xi)$ and $T(\xi)$. This set, unlike the set (16)–(19) of the foregoing problem, does not allow a simple decomposition into two independent problems for the functions $\Sigma(\xi)$ and $T(\xi)$.

Using the known relation between Riemann–Hilbert and Riemann problems (Gakhov, 1966), we write the system of boundary-value problems (47) in more convenient form of Riemann problems:

$$\Sigma^+(\xi) - \Sigma^-(\xi) = 0, \quad T^+(\xi) - T^-(\xi) = 0, \qquad \xi < l_* - ct,\ \xi > l_*, \tag{49}$$

$$\frac{n_1}{R}\Sigma^+(\xi) - \frac{\bar{n}_1}{\bar{R}}\Sigma^-(\xi) + \frac{q}{R}T^+(\xi) - \frac{\bar{q}}{\bar{R}}T^-(\xi) = 2\mu V_0'(\xi) \quad l_* - ct < \xi < l_*, \tag{50}$$

$$-\frac{q}{R}\Sigma^+(\xi) + \frac{\bar{q}}{\bar{R}}\Sigma^-(\xi) + \left(\frac{n_2}{R} - \frac{\mu}{fc}\right)T^+(\xi) - \left(\frac{\bar{n}_2}{\bar{R}} - \frac{\mu}{fc}\right)T^-(\xi) = 2\mu c^{-1}W_0(\xi), \quad l_* - ct < \xi < l_*. \tag{51}$$

We now multiply, e.g., Eq. (50) by a certain constant s and add it term-by-term to Eq. (51):

$$\frac{1}{R}(-q + sn_1)\left[\Sigma^+(\xi) + aT^+(\xi)\right] - \frac{1}{\bar{R}}(-\bar{q} + s\bar{n}_1)\left[\Sigma^-(\xi) + bT^-(\xi)\right] = 2\mu\left[sV_0'(\xi) - c^{-1}W_0(\xi)\right]. \tag{52}$$

Here

$$a = \frac{n_2 + sq - \mu R f^{-1}c^{-1}}{-q + sn_1}, \qquad b = \frac{\bar{n}_2 + s\bar{q} - \mu\bar{R}f^{-1}c^{-1}}{-\bar{q} + s\bar{n}_1}. \tag{53}$$

We choose s so that $a = b$. Then, from Eq. (53), we obtain the following equation for s

$$\frac{n_2 + sq - \mu R f^{-1}c^{-1}}{-q + sn_1} = \frac{\bar{n}_2 + s\bar{q} - \mu\bar{R}f^{-1}c^{-1}}{-\bar{q} + s\bar{n}_1},$$

which has a solution

$$s_n = \frac{1}{2}m_1^{-1}\left[p + (-1)^{n-1}\left(p^2 - 4m_1m_2\right)^{1/2}\right], \quad n = 1, 2, \tag{54}$$

where

$$m_j = -\bar{n}_j q + n_j\bar{q} + \mu f^{-1}c^{-1}(j-1)\left(\bar{R}q - R\bar{q}\right), \quad j = 1, 2,$$

$$p = -n_1\bar{n}_2 + \bar{n}_1 n_2 + \mu f^{-1}c^{-1}\left(\bar{R}n_1 - R\bar{n}_1\right). \tag{55}$$

Substituting the values s_1 and s_2 from Eq. (54) term-by-term into Eq. (53) and then, together with the corresponding values of a_1 and a_2 (since $a = b$) into Eq. (52), we obtain

$$\left[\Sigma^+(\xi) + a_nT^+(\xi)\right] - \beta_n\left[\Sigma^-(\xi) + a_nT^-(\xi)\right] = 2\nu_n\left[s_nV_0'(\xi) - c^{-1}W_0(\xi)\right], \quad n = 1, 2;\ l_* - ct < \xi < l_*, \tag{56}$$

where

$$a_n = \frac{n_2 + s_nq - \mu R f^{-1}c^{-1}}{-q + s_nn_1}, \quad \beta_n = \frac{R}{\bar{R}}\frac{-\bar{q} + s_n\bar{n}_1}{-q + s_nn_1}, \quad \nu_n = \frac{\mu R}{-q + s_nn_1}. \tag{57}$$

The conditions (49) can be written in the form, analogous to Eq. (56). For this purpose, we multiply the condition for Σ by a_n and add to the condition for T:

$$\left[\Sigma^+(\xi) + a_nT^+(\xi)\right] - \left[\Sigma^-(\xi) + a_nT^-(\xi)\right] = 0, \quad \xi < l_* - ct,\ \xi > l_*;\ n = 1, 2. \tag{58}$$

The conditions (56) and (58) represent two independent Riemann problems (for $n = 1$ and $n = 2$, respectively) for functions $\Sigma(\xi) + a_n T(\xi)$. The solutions of these problems which vanish at infinity can be written out for the case of discontinuous coefficients using the known formulae by Gakhov (1966). As a result, we obtain for $z_j = \xi + i0$

$$\Sigma^+(\xi) + a_n T^+(\xi) = \frac{\nu_n}{\pi i} G_n(\xi) \int_{l_*-ct}^{l_*} G_n^{-1}(\xi') \left[s_n V_0'(\xi') - c^{-1} W_0(\xi') \right] \frac{d\xi'}{\xi' - \xi} + \nu_n \left[s_n V_0'(\xi) - c^{-1} W_0(\xi) \right],$$

$$n = 1, 2. \tag{59}$$

Here

$$G_n(\xi) = \left(\frac{l_* - \xi}{l_* - ct - \xi} \right)^{\alpha_n}, \quad \alpha_n = -\frac{1}{2\pi i} \ln \beta_n, \quad n = 1, 2, \tag{60}$$

provided that

$$-\pi < \arg \beta_n < \pi. \tag{61}$$

Considering Eq. (59) as a set of two (for $n = 1, 2$) algebraic equations with respect to the functions $\Sigma(\xi)$ and $T(\xi)$, we find

$$\begin{bmatrix} \Sigma(\xi) \\ T(\xi) \end{bmatrix} = \sum_{n=1}^{2} \frac{(-1)^{n-1}}{a_2 - a_1} \nu_n \times \begin{bmatrix} a_3 - n \\ -1 \end{bmatrix} \left\{ \frac{1}{\pi i} G_n(\xi) \int_{l_*-ct}^{l_*} G_n^{-1}(\xi') \left[s_n V_0'(\xi') - c^{-1} W_0(\xi') \right] \frac{d\xi'}{\xi' - \xi} + s_n V_0'(\xi) - c^{-1} W_0(\xi) \right\}. \tag{62}$$

Now, the formulae (10), (22)–(24) and (62) give the formal expressions for stresses in the half-plane

$$\begin{bmatrix} \sigma_y(x, y, t) \\ \tau_{xy}(x, y, t) \end{bmatrix} = -\frac{1}{2\pi^2} \mathrm{Re} \int_{n=1}^{2} \frac{(-1)^{n-1} \nu_n(c)}{a_2(c) - a_1(c)} \frac{1}{R(c)} \times \left\{ \begin{bmatrix} \gamma^2(c) a_{3-n}(c) - 2i\gamma(c)\gamma_2(c) \\ -2i\gamma(c)\gamma_1(c) a_{3-n}(c) - 4\gamma_1(c)\gamma_2(c) \end{bmatrix} N_n(z_1(x, y, t, c), c) + \begin{bmatrix} 4\gamma_1(c)\gamma_2(c) a_{3-n}(c) - 2i\gamma(c)\gamma_2(c) \\ 2i\gamma(c)\gamma_1(c) a_{3-n}(c) - \gamma^2(c) \end{bmatrix} N_n(z_2(x, y, t, c), c) \right\} dc, \tag{63}$$

where

$$N_n(z_j) = G_n(z_j) \int_{l_*-ct}^{l_*} G_n^{-1}(\xi) \left[s_n V_0'(\xi) - c^{-1} W_0(\xi) \right] \frac{d\xi}{\xi - z_j}. \tag{63a}$$

6. Expressions for stresses in the contact zone

Substituting the expressions for $\Sigma(\xi)$ and $T(\xi)$, Eq. (62), into Eq. (34) and taking into account Eqs.

(60) and (9), we obtain

$$\begin{bmatrix} \sigma_y(x,0,t) \\ \tau_{xy}(x,0,t) \end{bmatrix} = -\frac{1}{2\pi^2}\mathrm{Re}\int_\Gamma \sum_{n=1}^{2} \frac{(-1)^{n-1}\nu_n(c)}{a_2(c)-a_1(c)} \begin{bmatrix} a_{3-n}(c) \\ -1 \end{bmatrix} \Bigg\{ \left(\frac{l_* - x + ct}{l_* - x}\right)^{\alpha_n(c)} \times \int_{l_*-ct}^{l_*} \left(\frac{l_*-\xi'}{l_*-ct-\xi'}\right)^{-\alpha_n(c)} \left[s_n(c)V_0'(\xi',c) - c^{-1}W_0(\xi',c)\right] \frac{\mathrm{d}\xi'}{\xi'-x+ct} + \pi i\left[s_n(c)V_0'(x,t,c) - c^{-1}W_0(x,t,c)\right]\Bigg\}\,\mathrm{d}c. \tag{64}$$

The integrands (in the integral with respect to c) in Eq. (64) are analytic in the c-plane outside the cuts $(-\infty, -c_j)$, (c_j, ∞), (c_*, ∞) and (c_{**}, ∞) along the axis $\mathrm{Im}\, c = 0$ and with a finite number of poles being excluded (c_* and c_{**} are the solutions of the Eq. (39)). Transforming the contour Γ along the axis $\mathrm{Im}\, c = 0$, we write the expression (64) in the resultant form

$$\begin{bmatrix} \sigma_y(x,0,t) \\ \tau_{xy}(x,0,t) \end{bmatrix} = -\frac{1}{2\pi^2}\mathrm{Re}\left(\int_{-c_1}^{0} + \int_0^{c_1}\right) \sum_{n=1}^{2} \Bigg\{ \frac{(-1)^{n-1}\nu_n(c)}{a_2(c)-a_1(c)} \begin{bmatrix} a_{3-n}(c) \\ -1 \end{bmatrix} \left(\frac{l_* - x + ct}{l_* - x}\right)^{\alpha_n(c)} \times \int_{l_*-ct}^{l_*} \left(\frac{l_*-\xi'}{l_*-ct-\xi'}\right)^{-\alpha_n(c)} \left[s_n(c)V_0'(\xi',c) - c^{-1}W_0(\xi',c)\right] \frac{\mathrm{d}\xi'}{\xi'-x+ct} + \pi i\left[s_n(c)V_0'(x,t,c) - c^{-1}W_0(x,t,c)\right]\Bigg\}\,\mathrm{d}c + \mu\left(\int_{-\infty}^{-c_1} + \int_{c_1}^{\infty}\right)\frac{R^{**}(c)}{S(c)} \times \int_{l_*-ct}^{l_*} \begin{bmatrix} \left(R^{**}(c)f^{-1}c^{-1} - n_2^*(c)\right)V_0'(\xi',c) - q^{**}(c)c^{-1}W_0(\xi',c) \\ q^{**}(c)V_0'(\xi',c) - n_1^*(c)c^{-1}W_0(\xi',c) \end{bmatrix} \frac{\mathrm{d}\xi'\mathrm{d}c}{\xi'-x+ct}. \tag{65}$$

Here

$$S = q^{**2} + n_1^* n_2^* - \mu R^{**} n_1^* f^{-1} c^{-1}, \quad R^{**} = \gamma^2 + 4\tau_1^*\gamma_2^*,$$

$$n_j^* = -\gamma_j^*\left(1-\gamma^2\right), \quad q^{**} = \gamma + 2\gamma_1^*\gamma_2^*, \quad \gamma_j^* = \left(c^2/c_j^2 - 1\right)^{1/2}, \tag{66}$$

and for $s_n(c)$, $a_n(c)$, $\nu_n(c)$ and $\alpha_n(c)$ in Eq. (65) we have from Eqs. (54), (55), (57) and (60):

For $0 < c < c_2$

$$s_n = \frac{\mu R}{2qfc}\left[1 + (-1)^{n-1}\sqrt{D}\right], \quad D = 1 - 4\frac{n_2}{n_1}\left(\frac{qfc}{\mu R}\right)^2,$$

$$a_n = \frac{n_2 + s_n q - \mu R f^{-1} c^{-1}}{q + s_n n_1}, \quad \nu_n = \frac{\mu R}{-q + s_n n_1},$$

$$\alpha_n = \theta_n = \frac{1}{\pi}\arctan\frac{\gamma_1\left(1-\gamma^2\right)s_n}{q}, \quad D \geq 0,$$

$$\alpha_n = \varepsilon_n = \frac{1}{2\pi}\arctan\frac{d\gamma_1\left(1-\gamma^2\right)}{q - g_n\gamma_1\left(1-\gamma^2\right)} + \frac{1}{2\pi}\arctan\frac{d\gamma_1\left(1-\gamma^2\right)}{q + g_n\gamma_1\left(1-\gamma^2\right)} - \frac{1}{2\pi i}\ln\left[\frac{\left(-q + g_n\gamma_1\left(1-\gamma^2\right)\right)^2 + d^2\gamma_1^2\left(1-\gamma^2\right)^2}{\left(q + g_n\gamma_1\left(1-\gamma^2\right)\right)^2 + d^2\gamma_1^2\left(1-\gamma^2\right)^2}\right]^{1/2}, \quad D < 0,$$

$$d = \mu R f^{-1} c^{-1}, \quad g_n = d(-1)^{n-1}\sqrt{-D}; \tag{67}$$

for $c_2 < c < c_1$

$$s_n = \frac{\mu\gamma^2 f^{-1} c^{-1} - n_2^*}{2\gamma_1\gamma_2^*}\left[1 + (-1)^{n-1}\sqrt{1 - 4\frac{i\gamma_1\gamma_2^*\left(2\mu\gamma^2 f^{-1}c^{-1} - \gamma_2^*\right)}{\left(\mu\gamma^2 f^{-1}c^{-1} - n_2^*\right)^2}}\right],$$

$$v_n = \frac{\mu R^*}{-q^* + s_n n_1}, \quad a_n = \frac{n_2^* + s_n q^* - \mu R^* f^{-1} c^{-1}}{q^* + s_n n_1},$$

$$\alpha_n = \delta_n = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{4\gamma_1\gamma_2^*}{\gamma^2} - \frac{1}{2}\arctan\frac{\rho_{nr}\gamma_1\left(1-\gamma^2\right) + 2\gamma_1\gamma_2^*}{\gamma - \rho_{ni}\gamma_1\left(1-\gamma^2\right)} + \frac{1}{2\pi}\arctan\frac{\rho_{nr}\gamma_1\left(1-\gamma^2\right) + 2\gamma_1\gamma_2^*}{\gamma + \rho_{ni}\gamma_1\left(1-\gamma^2\right)} - \frac{1}{2\pi i}\ln\left[\frac{\left(\gamma - \rho_{ni}\gamma_1\left(1-\gamma^2\right)\right)^2 + \left(\rho_{nr}\gamma_1\left(1-\gamma^2\right) + 2\gamma_1\gamma_2^*\right)^2}{\left(\gamma + \rho_{ni}\gamma_1\left(1-\gamma^2\right)\right)^2 + \left(\rho_{nr}\gamma_1\left(1-\gamma^2\right) + 2\gamma_1\gamma_2^*\right)^2}\right]^{1/2}, \tag{68}$$

where

$$R^* = \gamma^2 - 4i\gamma_1\gamma_2^*, \quad q^* = \gamma - 2i\gamma_1\gamma_2^*, \quad \rho_{nr} = \operatorname{Re} s_n, \quad \rho_{ni} = \operatorname{Im} s_n. \tag{69}$$

If a geometry of the rigid body is such that the function $l(t)$ is unknown, then it can be found from the condition (42a) which in this case is of the form

$$\operatorname{Re}\int_{-\infty}^{\infty}\sum_{n=1}^{2}\frac{\mu R(c)}{-q(c) + s_n(c)n_1(c)}\frac{(-1)^{n-1}a_{3-n}(c)}{a_2(c) - a_1(c)}\left\{\left(\frac{l_* - l(t) + ct}{l_* - l(t)}\right)^{\alpha_n(c)} \times \int_{l_*-ct}^{l_*}\left(\frac{l_* - \xi'}{l_* - ct - \xi'}\right)^{-\alpha_n(c)}\left[s_n(c)V_0'\left(\xi', c\right) - c^{-1}W_0\left(\xi', c\right)\right]\frac{\mathrm{d}\xi'}{\xi' - l(t) + ct} + \pi i\left[s_n(c)V_0'(l(t), t, c) - c^{-1}W_0(l(t), t, c)\right]\right\}\mathrm{d}c = 0. \tag{70}$$

This equation has a solution, i.e., the stresses are bounded (in fact, zero) at the point $x = l(t)$, only in the case where $\dot{l}(t) < c_{\mathrm{R}}$ or $\dot{l}(t) > c_1$. In the case where $c_{\mathrm{R}} < \dot{l}(t) < c_1$ the function $l(t)$ can be found from the expression for $v_0'(x, t)$ in Eq. (12) which after integration with respect to x gives the following equation for $l(t)$

$$\int_{-\infty}^{l(t)} \mathrm{Re}\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\mu R(c)}\big[n_1(c)\Sigma\big(\xi(x,t,c),c\big)+q(c)T\big(\xi(x,t,c),c\big)\big]\,\mathrm{d}c\,\mathrm{d}x = v_0'(l(t),t). \tag{71}$$

Here $\Sigma(\xi)$ and $T(\xi)$ have the form (62).

In the limit for $x\to l(t)-0$ with account for Eq. (44), we find

$$\begin{bmatrix}\sigma_y(x,0,t)\\ \tau_{xy}(x,0,t)\end{bmatrix} \simeq \mathrm{Re}\int_{\dot l(t)}^{\infty}\sum_{n=1}^{2}\begin{bmatrix}a_{3-n}(c)\\ -1\end{bmatrix}\left(\frac{c-\dot l(t)}{l(t)-x}t\right)^{\alpha_n(c)} K_n(t,c)\,\mathrm{d}c, \tag{72}$$

where

$$\begin{aligned}K_n(t,c) = &-\frac{1}{\pi^2}\frac{(-1)^{n-1}v_n(c)}{a_2(c)-a_1(c)}\left\{\int_{l(t)-ct}^{l(t)}\big(\xi'+ct-l(t)\big)^{\alpha_n(c)-1}\big(\xi'-l(t)\big)^{-\alpha_n(c)}\right.\\ &\left.\times\Big[s_n(c)V_0'\big(\xi',c\big)-c^{-1}W_0\big(\xi',c\big)\Big]\,\mathrm{d}\xi' + \pi i\big[s_n(c)V_0'(l(t),t,c)-c^{-1}W_0(l(t),t,c)\big]\right\}.\end{aligned} \tag{73}$$

It follows from Eqs. (71), (65), (67) and (68) that in the particular case where the rigid body (punch) has a differentiable boundary at the point of $x=l(t)$, the stresses have the following limiting behavior for $x\to l(t)$

$$\sigma_y \sim \tau_{xy} \sim \begin{cases} 0 & \text{for } \dot l(t) < c_\mathrm{R} \text{ and } D\big(\dot l(t)\big)\geq 0,\\ \sum_{n=1}^{2} A_n(l(t)-x)^{-\theta_n(\dot l(t))} & \text{for } c_\mathrm{R} < \dot l(t) < c_2 \text{ and } D\big(\dot l(t)\big)\geq 0,\\ \sum_{n=1}^{2} B_n(l(t)-x)^{-\varepsilon_{nr}(\dot l(t))-i\varepsilon_{ni}(\dot l(t))} & \text{for } 0 < \dot l(t) < c_2 \text{ and } D\big(\dot l(t)\big) < 0,\\ \sum_{n=1}^{2} C_n(l(t)-x)^{-\delta_{nr}(\dot l(t))-i\delta_{ni}(\dot l(t))} & \text{for } c_2 < \dot l(t) < c_1,\\ Av_0'(l(t),t)+Bw_0(l(t),t)\ \big(v_0'=\partial v_0/\partial x\big) & \text{for } c_1 < \dot l(t) < \infty \end{cases} \tag{74}$$

Here A_n, B_n, C_n, A and B are some bounded constants, $D(\dot l(t))$ and $\theta_n(\dot l(t))$ are of the form $D(c)$ and $\theta(c)$ from Eq. (67) for $c=\dot l(t)$, $\varepsilon_n(\dot l(t))=\varepsilon_{nr}(\dot l(t))+i\varepsilon_{ni}(\dot l(t))$ ($\varepsilon_{nr}=\mathrm{Re}\,\varepsilon_n$, $\varepsilon_{ni}=\mathrm{Im}\,\varepsilon_n$) and $\delta_n(\dot l(t))=\delta_{nr}(\dot l(t))+i\delta_{ni}(\dot l(t))$ ($\delta_{nr}=\mathrm{Re}\,\delta_n$, $\delta_{ni}=\mathrm{Im}\,\delta_n$) are of the form $\varepsilon_n(c)$ and $\delta_n(c)$ from Eqs. (67) and (68) for $c=\dot l(t)$. It is seen from Eq. (74) that the stresses have a oscillating singularity in the case $0<\dot l(t)<c_2$ for $D(\dot l(t))<0$ and in the case $c_2<\dot l(t)<c_1$, and the stresses change sign an infinite number of times at $x\to l(t)$. This lead to the violation of the condition $\sigma_y\leq 0$ in the conditions (3). Consequently, the conditions of viscous friction in the form (3) are acceptable (from the viewpoint of realization of the condition $\sigma_y\leq 0$) only in the case where $0<\dot l(t)<c_2$ and $D(\dot l(t))\geq 0$.

Note, that the oscillating singularities are typical for some elastostatic and elastodynamic problems. The typical examples are the problems of interfacial cracks in the Mode I or mixed Mode I–II cases (see e.g. Achenbach et al., 1976) and the contact problems with adhesion (see e.g. Muskhelishvili, 1953b).

7. Concluding remarks

The practical suitability is shown of the new method proposed by Shmegera (1997, 1998) to construct the exact solutions of non-automodeling initial boundary-value mixed problems of elastodynamics with the conditions of contact friction.

Here, we restrict our consideration to formal analysis of the obtained solutions, i.e., the existence, closure and consistency of solution. Certainly, these solutions require a more detailed analytical and numerical analysis, which will be a subject of a separate paper.

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Appendix A. A representation of solution of wave equation and its a relation with the Radon transform.

A.1. A two-dimensional Radon transform

A two-dimensional Radon transform $F(s, \omega)$ of some function $f(\mathbf{x})$ is determined (see, e.g., Ludvig, 1966; Helgason, 1980) as an integral along the line $\boldsymbol{\omega}\cdot\mathbf{x}=s$ ($\boldsymbol{\omega}=\{\omega_1,\omega_2\}$, $\mathbf{x}=\{x_1,x_2\}$) of the following form

$$F(s,\boldsymbol{\omega})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\delta(\boldsymbol{\omega}\cdot\mathbf{x}-s)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{A1}$$

where $\delta(\cdot)$ is Dirac delta-function. It is obvious that $F(s,\boldsymbol{\omega})$ is homogeneous function with the homogeneity degree equal to -1:

$$F(\alpha s, \alpha\boldsymbol{\omega})=|\alpha|^{-1}F(s,\boldsymbol{\omega}), \tag{A2}$$

and $F(s,\boldsymbol{\omega})$ is an even function. It follows from Eq. (A1) that $f(\mathbf{x})=0$ for $r>0$ ($r=\sqrt{x_1^2+x_2^2}$) then $F(s,\boldsymbol{\omega})=0$ for $|s|>a$.

An inverse formulae of Radon transform (A1) can be written in the following form

$$f(\mathbf{x})=-\frac{1}{4\pi^2}\int_{\Gamma}\int_{-\infty}^{\infty}\frac{\partial F(s,\omega_1,\omega_2)}{\partial s}\frac{\mathrm{d}s(\omega_1\mathrm{d}\omega_2-\omega_2\mathrm{d}\omega_1)}{s-\omega_1x_1-\omega_2x_2}, \tag{A3}$$

where Γ is an arbitrary contour in the plane ω.

For a fixed value of ω, the integrand in Eq. (A3) (in the integral with respect to ω) is orthogonal to the ω-plane. In the other words, the integrand of inverse formula is a plane wave. From this point of view, the formula (A3) can be considered as a continuous expansion of function $f(\mathbf{x})$ into the plane waves of arbitrary form.

A.2. A representation of the solution of two-dimensional wave equation

This represents the solution of two-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2}, \tag{A4}$$

in the form of continuous (integral) superposition of some arbitrary functions

$$u(x, y, t) = \frac{1}{2\pi i}\int_{\Gamma} F(z(x, y, t, c), c)\,\mathrm{d}c, \tag{A5}$$

where $F(z)$ is an arbitrary, twice differentiable (or analytic, if z is complex) function. Γ is an arbitrary contour in the complex plane c. Substituting Eq. (A5) into (A4), we find that the expression (A5) is the solution of wave equation (A4) if and only if the function z satisfies the following equations

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c_0^2}\frac{\partial^2 z}{\partial t^2}, \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{c_0^2}\left(\frac{\partial z}{\partial t}\right)^2. \tag{A6}$$

It is well known (see, for example, Smirnov, 1974) that the general solution of system (A6) determines from the equation

$$k(z) = l(z)t + m(z)x + n(z)y, \tag{A7}$$

in which the functions l, m and n are related by

$$l^2(z) = c_0^2\left[m^2(z) + n^2(z)\right]. \tag{A8}$$

Consider a simple case, most interesting for the expression (A5). Assume the functions l, m and n to be constant, and the function k to be equal to z. Then, the function $F(z)$ is the plane wave for the real l, m and n. If l, m and n are the complex values with a variable argument, then from the physical point of view, the function $F(z)$ is not the plane wave. This function $F(z)$ for the complex z can be called as a complex plane wave.

Denoting $l = -c$ and $m = -1$, we find from Eq. (A8) that $n = \pm i(1 - c^2/c_0^2)^{1/2}$ where $i = \sqrt{-1}$. For selection of the uniquely branch of radical $(1 - c^2/c_0^2)^{1/2}$ in the complex plane c, we make the cuts $(-\infty, -c_0)$ and (c_0, ∞) along the axis $\mathrm{Im}\, c = 0$ and fix this branch by the condition $(1 - c^2/c_0^2)^{1/2} > 0$ for $\mathrm{Im}\, c = 0$. In this case the expression for z becomes

$$z = x - ct + iy\sqrt{1 - c^2/c_0^2}, \tag{A9}$$

or, in the terms of following notations

$$\xi = x - ct, \quad \eta = \gamma y, \quad \gamma = \sqrt{1 - c^2/c_0^2}, \tag{A10}$$

in the form

$$z = \xi + i\eta. \tag{A11}$$

In the case under consideration and provided that $c < c_0$ the function $F(z)$, in the terms of ξ and η, satisfies the Laplace's equations

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} = 0. \tag{A12}$$

In the case $c > c_0$, the Eq. (A12) reduce to the one-dimensional wave equation.

Since the real and imaginary parts of Eq. (A5) satisfy Eq. (A4), the solution can be selected in the form of real part of representation (A5):

$$u(x, y, t) = \mathrm{Re}\frac{1}{2\pi i}\int_{\Gamma} F\left(x - ct + iy\sqrt{1 - c^2/c_0^2}, c\right) \mathrm{d}c. \tag{A13}$$

Let, now, $u_0(x, t)$ is a value of $u(x, y, t)$ for $y = 0$:

$$u(x, 0, t) = \mathrm{Re}\frac{1}{2\pi i}\int_{\Gamma} F^{+}\big(\xi(x, t, c), c\big) \,\mathrm{d}c = u_0(x, t). \tag{A14}$$

Here $F^{+}(z)$ is the boundary value of $F^{+}(\xi)$, which is analytic in the half-plane $\mathrm{Im}\, z > 0$. By using the relation $F^{-}(\xi) = -\bar{F}^{+}(\xi)$ where the overbar on $F^{+}(\xi)$ denotes the complex conjugate, we rewrite Eq. (A14) in the form

$$\frac{1}{4\pi i}\int_{\Gamma}\left[F^{+}\big(\xi(x, t, c), c\big) + F^{-}\big(\xi(x, t, c), c\big)\right] \mathrm{d}c = u_0(x, t). \tag{A15}$$

Now, we use Plemelj's formulae (see, for example, Muskhelishvili, 1953a) to rewrite the expression under the integral sign in Eq. (A12) as

$$F^{+}(\xi) + F^{-}(\xi) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{U_0(\xi')}{\xi' - \xi} \,\mathrm{d}\xi', \tag{A16}$$

where $U_0(\xi)$ is the function which satisfies the Hölder condition (see, e.g., Gakhov, 1966) in all the points (including infinity point). Substituting Eq. (A16) into Eq. (A15), we obtain

$$-\frac{1}{4\pi^2}\int_{\Gamma}\int_{-\infty}^{\infty} \frac{U_0(\xi', c)}{\xi' - \xi(x, t, c)} \,\mathrm{d}\xi' \,\mathrm{d}c = u_0(x, t). \tag{A17}$$

If the function $U_0(\xi, c)$ is derivative (with respect to ξ) of the two-dimensional Radon transform of function $u_0(x, t)$ (see the formulae (A1) and (A2)), i.e., if

$$U_0(\xi, c) = \frac{\partial}{\partial \xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x, t)\delta(x - ct - \xi) \,\mathrm{d}x \,\mathrm{d}t, \tag{A18}$$

then the expression (A16) and, consequently, (A15) can be considered as the inverse formulae of two-dimensional Radon transform (see the formula (A3) for $\mathbf{x} = \{x, t\}$ and $\omega = \{1, -c\}$).

Observe that application of Radon transform to the expression (A14) gives

$$\mathrm{Re}\, F^{+}(\xi) = U_0(\xi), \tag{A19}$$

i.e., transformation of left-hand side of Eq. (A14) actually eliminates the operation of integration with respect to c.

It should be noted that in the expressions (A17) and (A18) (from consideration of convenience) we somewhat depart from the formal separation of operations, accepted in the literature (see Eqs. (A1) and (A3)), where the integral part of formula (A18) is called the Radon transform, and the operation of differentiation is introduced in the inverse formulae. Nevertheless, we will call the expression (A18) as the Radon transform.

A.3. Some properties of the carrier of Radon transform

Let the function $f(x, t)$ has the following values depending on the intervals of change of variables x and t

$$f(x,t)=\begin{cases} f_0(x,t) & (t<0,\ -\infty<x<\infty),\\ f_1(x,t) & (t>0,\ -\infty<x<l(t)),\\ f_2(x,t) & (t>0,\ l(t)<x<\infty), \end{cases} \tag{A20}$$

then the Radon transform of the function $f(x, t)$ has the form

$$\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\begin{array}{l} f_0(x,t')H(-t)\\ f_1(x,t')H(t')H(t-t')H(l(t')-x)\\ f_2(x,t')H(t'-t)H(x-l(t')) \end{array}\right\}\delta\big(x-ct'-\xi\big)\,\mathrm{d}x\,\mathrm{d}t'$$

$$=\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\left\{\begin{array}{l} f_0(ct'+\xi,t')H(-t')\\ f_1(ct'+\xi,t')H(t')H(t-t')H\big(l(t')-ct'-\xi\big)\\ f_2(ct'+\xi,t')H(t'-t)H\big(ct'+\xi-l(t')\big) \end{array}\right\}\mathrm{d}t'$$

$$=\begin{cases} F_0(\xi) & (\xi>0)\ \text{for}\ t<0,\\ F_1(\xi) & (l_*-ct<\xi<l_*),\\ F_2(\xi) & (\xi<l_*-ct). \end{cases} \tag{A21}$$

Here $H(\ldots)$ is the Heaviside function and $l_*=l(t_*)$ where t_* is a solution of equation

$$l(t_*)-ct_*-\xi=0, \tag{A22}$$

the left-hand side of which is the argument of Heaviside function in Eq. (A21).

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